Closed-form solutions for piezomagnetic inhomogeneities embedded in a non-piezomagnetic matrix

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Received 8 September 2003; accepted 1 July 2004
Available online 10 August 2004

Abstract
This paper presents two different analytical methods to investigate the magneto-mechanical coupling effect for piezomagnetic inhomogeneities embedded in a non-piezomagnetic matrix. First, the magnetoelastic solution is expressed in terms of magnetoelastic Green’s function that can be decoupled into elastic Green’s function and magnetic Green’s function. Second, the problem is analyzed by the equivalent inclusion method, and then, the formulation of the inhomogeneity problem can be decoupled into an elastic problem and a magnetic inhomogeneity problem connected by some eigenstrain and eigenmagnetic fields. For the piezomagnetic composites with a non-piezomagnetic matrix, these two solutions are completely equivalent each other though they are obtained by means of two different methods. Moreover, based upon the unified energy method, the effective magnetoelastic moduli of the composites are expressed explicitly in terms of phase properties and volume fractions. Then the dilute and Mori–Tanaka schemes are discussed, respectively. Finally, the calculations are made to predict the effective magnetoelastic moduli and illustrate the performance of each model.

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Keywords: Piezomagnetic; Green’s function; Effective inclusion

1. Introduction
The composites, composed of ferromagnetic particles dispersed in a non-ferromagnetic matrix such as a polymer susceptible to elastic deformations, constitute a very interesting group of magnetic materials, which can take the advantage of each constituent and consequently have superior magnetoelastic effect as compared to conventional piezomagnetic materials (Bednarek, 1998a, 1998b, 1998c, 1999a, 1999b, 2000). Because of a unique property in terms of magnetoelastic coupling effects resulting from the magneto-mechanical interaction, the ferromagnetic composites attract great attentions for the design of intelligent or active structures. A remarkably wide variety of applications includes the development of magneto-mechanical transducers, magnetic sensors, strain gages, ultrasonic generators, and stress sensors (Karl et al., 2000; Shim et al., 1998). These applications require a sufficiently big magnetostriction and are sensitive to stress. Fortunately, most of the ferromagnetic composites with the elastic matrix susceptible to deformation provide the excellent magnetostrictive function. This may explain that the piezomagnetic materials play an important role in smart or intelligent structures.

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For elastic solids and piezoelectric composites, the Green’s functions method and the equivalent inclusion method are so explicit and efficient that they are usually employed to investigate the inclusion and inhomogeneity problems (Duun, 1993, 1994; Duun and Taya, 1993; Mura, 1987; Fan and Qin, 1995; Wang, 1992; Jiang, 1998). Recently, the piezoelectric-piezomagnetic composites attract increasing attentions for the magneto-electro-elastic coupling effect while their solutions are not explicit yet (Huang, 1997, 1998, 2000; Wu and Huang, 2000; Liu, 2001). Li and Duun (Li and Dunn, 1998; Li, 2000) studied magnetoelastic multi-inclusion and inhomogeneity problems and their applications in composite materials. However, theoretical studies are not sufficient to understand the coupling between magnetic filed and mechanical loading in ferromagnetic composites with the elastic matrix. On the other hand, from recently increasing emphasis on the practical development of many new ferromagnetic composites for smart or intelligent structures such as Terfenol-D composites with polymer matrices, it is necessary to develop some theories to guide their application.

One interest in micromechanics analysis of the ferromagnetic composites is related to the prediction of the effective magnetostriction. Herbst et al. (1997) developed a simple model to estimate the effective magnetostriction of a composite. Based on the Green function, Nan (Nan, 1987; Nan and Weng, 1999; Nan et al., 2000) proposed a model to analysis the effective magnetostriction of magnetostrictive composites. Chen et al. (1999) studied the elastic modulus of the matrix on magnetostrictive strain in composites. Duenas and Carman (2000) investigated large magnetostrictive response of Terfenol-D resin composite. Wan et al. (2004) modified the double-inclusion model to study the permeability dependence of the effective magnetostriction of magnetostrictive Composites. The authors (Feng et al., 2002, 2003) also employed the double-inclusion method to predict effective magnetostriction and moduli of magnetostrictive composites.

As employed in practice, the ferromagnetic composite considered in this paper consists of the ferromagnetic inclusions and a matrix that is a non-piezomagnetic material. For such kind of the composite, the formulation will be simplified because the magneto-mechanical coupling in the matrix is absent. However, the simplified model is more accessible and reasonable to practical inhomogeneity problems since most of ferromagnetic composites have a polymer matrix that is non-piezomagnetic. Two different analytical methods are presented to investigate the magneto-mechanical coupling effect for piezomagnetic inhomogeneities embedded in a non-piezomagnetic matrix. First, the magnetoelastic solution is directly expressed in terms of magnetoelectric Green’s functions that can be decoupled into the elastic Green’s function and the magnetic Green’s function. Second, the problem is analyzed by the equivalent inclusion method. According to the assumption of no magneto-mechanical coupling in the matrix, the original piezomagnetic inhomogeneity problem can also be partially decoupled into two inhomogeneity problems: one is elastic and the other is magnetic. These two inhomogeneity problems can be linked by some eigen-quantities that correspond to magneto-mechanical coupling terms inside the piezomagnetic inhomogeneity. Therefore, in this connection, the Green’s functions corresponding to the coupled piezomagnetic materials are needless, while the Eshelby’s elastic solution and the Eshelby’s magnetic-type solution in a magnetic material play the important roles. For the piezomagnetic composites with a non-piezomagnetic matrix, these two solutions are completely equivalent each other though they are obtained in terms of two different methods.

To obtain the effective magnetoelastic moduli of the composites, the analytical model is proposed by a unified energy method. The effective magnetoelastic moduli of the composites are expressed explicitly in terms of phase properties and their volume fraction. Then the dilute and Mori–Tanaka schemes are discussed, respectively. Finally, the calculations are made to predict the effective magnetoelastic moduli and illustrate the performance of each model.

2. Green’s function method

The constitutive equations for a linear piezomagnetic inhomogeneity are

\[
\sigma_{ij} = C_{ijkl}^p (u_{k,l} - \epsilon_{kl}^p) + d_{mij}^p (\phi_m + H_m^p) \quad \text{in } \Omega, \tag{1}
\]

\[
B_i = d_{ijkl}^m (u_{k,l} - \epsilon_{kl}^m) - \mu_{im}^p (\phi_m + H_m^p) \quad \text{in } \Omega, \tag{2}
\]

where \(C_{ijkl}^p\) denotes elastic moduli, \(d_{mij}^p\) is the piezomagnetic coefficient, \(\mu_{im}^p\) is the magnetic permeability, \(B_i\) is the magnetic induction, \(H_i\) is the magnetic field, \(\epsilon_{kl}^p\) is the eigen-strain, and \(H_m^p\) is the eigen-magnetic-field. The superscript "\(p\)" refers to the material property and eigen-quantities of the piezomagnetic inhomogeneity. Note that the spontaneous magnetization \(M_0^p\) has the clearly physical meaning. Actually, the eigen-magnetic-field \(H_m^p\) is corresponding to the spontaneous magnetization \(M_0^p\) of the materials. Furthermore, \(M_0^p = \chi_{ij}^p H_i^p\), where \(\chi_{ij}^p\) are the susceptibility coefficients. In addition, \(u_i\) is displacement and \(\phi\) is magnetic potential energy, which is subjected to

\[
H_m = -\phi_m, \tag{3}
\]

\[
C_{ijkl}^p u_{k,l} = C_{ijkl}^p \xi_{kl}^p. \tag{4}
\]
Because the matrix is non-piezomagnetic, the constitutive equations for the matrix are

\[ \sigma_{ij} = C_{ijkl} u_{k,l} \quad \text{in} \quad D - \Omega, \quad (5) \]
\[ B_i = -\mu_{im} \phi_m \quad \text{in} \quad D - \Omega, \quad (6) \]

where \( C_{ijkl} \) is the elastic moduli of the matrix, \( \mu_{ij} \) is the magnetic permeability of the matrix.

If the characteristic function defined by

\[ h(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & \text{other} \end{cases} \]

is introduced, then the elastic, piezomagnetic and magnetic-constant tensors of the inhomogeneity can be written as

\[ C^* = C + e h(x), \quad (8) \]
\[ d^* = d^* h(x), \quad (9) \]
\[ \mu^* = \mu + \kappa h(x), \quad (10) \]

where

\[ e = C^* - C, \quad (11) \]
\[ \kappa = \mu^* - \mu. \quad (12) \]

According to Eqs. (8)–(12), (1), (2), (5), (6) can be rewritten as

\[ \sigma_{ij} = C_{ijkl} u_{k,l} + [e_{ijkl} u_{k,l} + d_{mlj}^* \phi_m - (e_{ijkl}^* v_{kl}^* - d_{mlj}^* H_m^*)] h(x), \quad (13) \]
\[ B_i = -\mu_{im} \phi_m + [d_{ikl}^* u_{k,l} - \kappa_{im} \phi_m - (d_{ikl}^* v_{kl}^* + \mu_{im}^* H_m^*)] h(x). \quad (14) \]

The quasi-static equilibrium equations for both the inhomogeneity and the matrix are

\[ \sigma_{ij,j} = 0, \quad (15) \]
\[ B_{i,i} = 0. \quad (16) \]

Substituting Eqs. (13) and (14) into Eqs. (15) and (16) yields

\[ C_{ijkl} u_{k,l,j} = -[e_{ijkl} u_{k,l} + d_{mlj}^* \phi_m - (C_{ijkl}^* v_{kl}^* - d_{mlj}^* H_m^*)] h(x), \quad (17) \]
\[ \mu_{im} \phi_{m,i} = [d_{ikl}^* u_{k,l} - \kappa_{im} \phi_m - (d_{ikl}^* v_{kl}^* + \mu_{im}^* H_m^*)] h(x). \quad (18) \]

Because there is no magneto-mechanical coupling in the matrix, the Green’s functions can be introduced as follows:

\[ C_{ijkl} G_{kp,lj}^\sigma (x - x') = -\delta_{kp} \delta(x - x'), \quad (19) \]
\[ \mu_{im} G_{im}^\phi (x - x') = \delta(x - x'). \quad (20) \]

In Eq. (19), the Green’s function is the elastic Green’s function defined by

\[ G_{ij}^\sigma (x - x') = (2\pi)^{-3} \int_{-\infty}^{+\infty} N_{ij}^\sigma (\xi) D^{-1} (\xi) \exp[i \xi \bullet (x - x')] d\xi, \quad (21) \]

where \( K_{ij} = C_{ijkl} \xi_j \xi_l \), \( N_{ij}^\sigma (\xi) \) is the cofactor of \( K_{ij} \), and \( D(\xi) \) is the determinant of \( K_{ij} \).

In Eq. (20), the Green’s function is the magnetic Green’s function defined by

\[ G^\phi (x - x') = -(2\pi)^{-3} \int_{-\infty}^{+\infty} (\mu_{ij} \xi_j \xi_l)^{-1} \exp[i \xi \bullet (x - x')] d\xi. \quad (22) \]

According to Eqs. (21) and (22), (17) and (18) can be written as
and the magnetic Eshelby tensor can be defined as

\[ \varepsilon_{ij} = \frac{1}{2} (u_{ij} + u_{ji}) = \varepsilon_{ij}^0 + \frac{1}{2} \int G_{\mu ij}(x - x') \, \varepsilon_{ij}^0 \, dx' \]

\( (1 + S : C^{-1} : \varepsilon) : \varepsilon^0 + S : C^{-1} : (d^s)^T \cdot H^0 + S : C^{-1} : [C^r : \varepsilon^* - (d^s)^T \cdot H^0] \).

\( (i + s : \mu^{-1} \cdot \kappa) \cdot H^0 = \mu^0 \cdot \varepsilon^0 + \mu^0 \cdot \varepsilon^* + \mu^0 \cdot H^0 \).

The elastic Eshelby tensor can be defined by

\[ S_{ijkl} = -\frac{1}{2} \int C_{mnkl} \{ G_{\mu mn}^0 (x - x') + G_{\mu mn}^0 (x - x') \} \, dx' \]

and the magnetic Eshelby tensor can be defined as

\[ s_{ij} = \int \mu_{pj} G_{\phi p}^0 (x - x') \, dx' \].

Then Eqs. (27) and (28) can be rewritten as:

\[ (I + S : C^{-1} : \varepsilon) : \varepsilon^0 + S : C^{-1} : (d^s)^T : H^0 + S : C^{-1} : [C^r : \varepsilon^* - (d^s)^T : H^0] \]

\[ (i + s : \mu^{-1} : \kappa) \cdot H^0 = \mu^0 \cdot \varepsilon^0 + \mu^0 \cdot \varepsilon^* + \mu^0 \cdot H^0 \]

where \( I \) and \( i \) are four-order and two-order identity tensors, respectively. \( (d^s)^T = d^s \), and \( (\cdot)^{-1} \) represents inverse. The superscript “T” refers to the variables and quantities of the piezomagnetic inhomogeneity.

In terms of Eqs. (31) and (32), the formulations for the quantities of the piezomagnetic inhomogeneity can be obtained as follows:

\[ \varepsilon^1 = N^1 : \varepsilon^0 + N^2 : H^0 + S^1 : \varepsilon^* + S^2 : H^*, \]

\[ H^1 = N^3 : \varepsilon^0 + N^4 : H^0 + S^3 : \varepsilon^* + S^4 : H^*, \]

\[ \]
and
\[ A^c = (I + S : C^{-1} : e)^{-1}, \quad (43) \]
\[ B^c = (i + s \cdot \mu^{-1} \cdot \kappa)^{-1}, \quad (44) \]
\[ \alpha = A^c : S : C^{-1} : (d^*)^T, \quad (45) \]
\[ \beta = B^c \cdot s \cdot \mu^{-1} \cdot d^*, \quad (46) \]
\[ \zeta = A^c : S : C^{-1} : C^*, \quad (47) \]
\[ \eta = B^c \cdot s \cdot \mu^{-1} \cdot \mu^*, \quad (48) \]

in which \( A^c \) is the elastic strain concentration tensor and \( B^c \) the magnetic concentration tensor. Eqs. (35), (36), (39), (40) can also be written as:
\[ N_1 = A^c + \alpha \cdot N^3, \quad (49) \]
\[ N_3 = \alpha \cdot N^4, \quad (50) \]
\[ S_1 = \zeta + \alpha \cdot S^3, \quad (51) \]
\[ S_3 = -\alpha \cdot (i - S^4). \quad (52) \]

Eqs. (35)–(52) are used to get magneto-elastic Eshelby tensor, elastic Eshelby tensor, and magnetic Eshelby tensor, respectively. Because the elastic Eshelby tensor and the magnetic Eshelby tensor have been obtained, Eqs. (33) and (34) are the closed-form solution for the strain field and the magnetic field in the inhomogeneity. \( S \) is not Voigt symmetric, nor \( N_1 \) and \( S_1 \).

For a magneto-mechanical coupling problem, the parameters \( \alpha \) and \( \beta \) are very important. \( \alpha \) is a three-order tensor that is a function of the shape and direction of the inhomogeneities, the elastic properties, and the piezomagnetic properties of the inhomogeneities and the matrix. And \( \beta \) is also a three-order tensor that is the function of the shape and direction of the inhomogeneities, the magnetic properties, and the piezomagnetic properties of the inhomogeneities and the matrix. When the inhomogeneity is non-magneto-mechanical \((d^* = 0)\), \( \alpha \) and \( \beta \) are zero, then \( N_1 \), \( N_3 \), \( S_2 \) and \( S_3 \) are zero too. Therefore, \( N^1, N^4, S_1, \) and \( S_4 \) reduce to \( A^c \) that is the elastic strain concentration tensor, \( B^c \) that is the magnetic concentration tensor, \( \zeta \) and \( \eta \), respectively. Thus, the problem can be simplified to an elastic inhomogeneity and a magnetic inhomogeneity problem. If the material is homogeneous, \( N_1 \) and \( N^4 \) can reduce to the four-order and two-order identity tensor, respectively. \( S_1 \) reduces to the elastic Eshelby tensor \( S \) and \( S^4 \) to the magnetic Eshelby tensor \( s \).

3. Equivalent inclusion method

Since there is no coupling between the elastic fields and the magnetic fields in the matrix, the original piezomagnetic inhomogeneity problem can be partially decoupled into the following two equivalent inclusion problems.

3.1. The elastic equivalent inclusion problem

The stress in the inclusion with the eigen-strain \( e^* \) and the eigen-magnetic-field \( H^* \) under the uniform far field loading \( e^0 \) and \( H^0 \) is
\[ \sigma^I = \sigma^0 + \sigma. \quad (51) \]
From the linear constitutive equations, Eq. (51) can be rewritten as
\[ \sigma^I = C^* : (e - e^* + e^0 - e^H), \quad (52) \]
where
\[ C^* : e^H = (d^*)^T(H - H^* + H^0). \quad (53) \]
By use of the equivalent inclusion method, Eq. (52) can be rewritten as
\[ \sigma^I = C : (e + e^0 - e^{**}), \quad (54a) \]
where \( e^{**} \) is the total eigenstrain defined by
\[ C^* : (e - e^* + e^0 - e^H) = C : (e + e^0 - e^{**}). \quad (54b) \]
From the Eshelby inclusion solution, there is
\[ \varepsilon = S : \varepsilon^*, \]  
where \( S \) is the elastic Eshelby tensor. By substituting Eq. (54c) into Eq. (54b), the total eigen-strain can be expressed as
\[ \varepsilon^* = -[(C^* - C) : S + C]^{-1} : \left[ (C^* - C) : e^0 - C^* : e^H - C^* : e^s \right]. \]

3.2. The equivalent inclusion problem in a magnetic material

The magnetic induction in the inclusion with the eigen-strain \( e^s \) and the eigen-magnetic-field \( H^* \) under the uniform far field loading \( e^0 \) and \( H^0 \) is
\[ B^I = B^0 + B. \]

From the linear constitutive equations, Eq. (56) can be rewritten as
\[ B^I = \mu^* \cdot (H^0 + H - H^* - H^*), \]

where
\[ -\mu^* \cdot H^s = d^s : (e^0 + e - e^s). \]

By use of the equivalent inclusion method, Eq. (57) can be rewritten as
\[ B^I = \mu \cdot (H^0 + H - H^{**}), \]

where \( H^{**} \) is the total eigen-magnetic-field.

By means of the same way given in the above section, the total eigen-magnetic-field can be expressed as
\[ H^{**} = -[(\mu^* - \mu) \cdot s + \mu]^{-1} : \left[ (\mu^* - \mu) : H^0 - \mu^* : H^s - \mu^* : H^* \right] \]

in which \( s \) is the magnetic Eshelby tensor. Finally the total eigenstrain and eigen-magnetic-field can be written as
\[ e^{**} = D^1 : e^0 + D^2 : H^0 + D^3 : e^s + D^4 : H^*, \]
and
\[ H^{**} = [\kappa \cdot s + \mu + d^s : S : (e : S + C)^{-1} : (d^s)^T : s]^{-1} : [-(d^s - d^s : S : (e : S + C)^{-1} : e) : e^0 \]
\[ - \left[ \kappa + d^s : S : (e : S + C)^{-1} : (d^s)^T : H^0 + (d^s - d^s : S : (e : S + C)^{-1} : (e + C) : e^s \right] \]
\[ + [\kappa + \mu + d^s : S : (e : S + C)^{-1} : (d^s)^T : H^*]. \]

Eq. (62) can be written in a simple form as
\[ H^{**} = D^5 : e^0 + D^6 : H^0 + D^7 : e^s + D^8 : H^*, \]

where
\[ e = C^* - C, \quad d^0 = d^s - d = d^s, \quad \kappa = \mu^* - \mu. \]

Therefore, the strain and magnetic field in the inclusion can be written as
\[ e^I = S : e^{**} + e^0, \]
\[ H^I = s \cdot H^{**} + H^0, \]

Substituting Eqs. (61a) and (62a) into (64) and (65) leads to
\[ e^I = (S : D^I + I) : e^0 + S : D^2 : H^0 + S : D^3 : e^s + S : D^4 : H^*, \]
\[ H^I = s \cdot D^5 : e^0 + (s : D^6 + i) \cdot H^0 + s \cdot D^7 : e^s + s \cdot D^8 : H^*. \]
Then, Eqs. (66) and (67) can be expressed in a simple form as
\[
\varepsilon^I = N^1 : \varepsilon^0 + N^2 : \mathbf{H}^0 + S^1 : \varepsilon^* + S^2 : \mathbf{H}^*, \\
\mathbf{H}^I = N^3 : \varepsilon^0 + N^4 : \mathbf{H}^0 + S^3 : \varepsilon^* + S^4 : \mathbf{H}^*,
\]
(68)
(69)
The tensors are defined as follows:
\[
\mathbf{A}^c = (I + S : C^{-1} : \mathbf{e})^{-1} - 1, \\
\mathbf{B}^c = (i + s \cdot \mu^{-1} \cdot \mathbf{e})^{-1} - 1, \\
\alpha = \mathbf{A}^c : S : C^{-1} : (\mathbf{d}^*)^T, \\
\beta = \mathbf{B}^c \cdot s \cdot \mu^{-1} \cdot \mathbf{d}^*, \\
\xi = \mathbf{A}^c : S : C^{-1} : C^*, \\
\eta = \mathbf{B}^c \cdot s \cdot \mu^{-1} \cdot \mathbf{\mu}^*.
\] (70)

where
\[
N^1 = (I + \alpha \cdot \beta)^{-1} : \mathbf{A}^c, \quad (71a) \\
N^2 = (I + \alpha \cdot \beta)^{-1} : \alpha \cdot \mathbf{B}^c, \quad (71b) \\
N^3 = -(i + \beta : \alpha)^{-1} : \beta : \mathbf{A}^c, \quad (71c) \\
N^4 = (i + \beta : \alpha)^{-1} : \mathbf{B}^c, \quad (71d) \\
S^1 = I - (I + \alpha \cdot \beta)^{-1} : (I - \xi), \quad (71e) \\
S^2 = -(I + \alpha \cdot \beta)^{-1} : \alpha : (i - \eta), \quad (71f) \\
S^3 = (i + \beta : \alpha)^{-1} : \beta : (I - \xi), \quad (71g) \\
S^4 = i - (i + \beta : \alpha)^{-1} : (i - \eta). \quad (71h)
\]

It is interesting to note that the results of the inhomogeneity by employing the equivalent inclusion method are exactly same as those obtained by use of the Green’s function approach.

4. Effective magnetoelastic properties

To predict all of the elastic, piezomagnetic and magnetic constants of a piezomagnetic composite, the analytical model is proposed by a unified energy method in this section. It is considered that a sufficiently large two-phase composite, \( D \), consists of randomly oriented ellipsoidal inhomogeneities \( \Omega = \Omega_1 + \Omega_2 + \cdots + \Omega_N \). The matrix with the non-piezomagnetic properties is denoted by \( D - \Omega \). Then the nth inhomogeneity’s constitutive equation is
\[
\sigma_{ij} = C_{ijkl}^m \varepsilon_{kl} - d_{mij}^n H_m \quad \text{in } \Omega_n, \\
B_i = d_{ikj}^n \varepsilon_{kl} + \mu_{im}^n H_m \quad \text{in } \Omega_n,
\]
(72a)
(72b)
where \( C_{ijkl}^m \) denotes elastic moduli, \( d_{mij}^n \) is the piezomagnetic coefficient and \( \mu_{im}^n \) the magnetic permeability of the nth inhomogeneity. The matrix’s constitutive equation is
\[
\sigma_{ij} = \overline{C}_{ijkl} \varepsilon_{kl} \quad \text{in } D - \Omega, \\
B_i = \overline{\mu}_{im} H_m \quad \text{in } D - \Omega,
\]
(73a)
(73b)
where \( \overline{C}_{ijkl} \) denotes elastic moduli, and \( \overline{\mu}_{ij} \) is the magnetic permeability of the matrix.

The piezomagnetic composite’s constitutive equation is
\[
\sigma_{ij} = \overline{C}_{ijkl} \varepsilon_{kl} - \ddot{d}_{mij}^n H_m, \\
B_i = \ddot{d}_{ikj}^n \varepsilon_{kl} + \dot{\mu}_{im} H_m, 
\]
(74a)
(74b)
where \( \overline{C}_{ijkl} \) denotes elastic moduli, \( \ddot{d}_{mij}^n \) is the piezomagnetic coefficient, and \( \dot{\mu}_{ij} \) the magnetic permeability of the composite.
Comparing Eq. (80) with Eq. (81) results in the following equation:

\[ G(\varepsilon_{ij}, H_i) = G^1(\varepsilon_{ij}, H_i) - G^2(\varepsilon_{ij}, H_i), \]  

where

\[ G^1(\varepsilon_{ij}, H_i) = \frac{1}{2V} \int_{V} \varepsilon_{ij} \sigma_{ij}(\varepsilon_{ij}, H_i) \, dV, \]  

\[ G^2(\varepsilon_{ij}, H_i) = \frac{1}{2V} \int_{V} H_i B_i(\varepsilon_{ij}, H_i) \, dV. \]

According to the constitutive equation (74a,b), the Gibbs free energy can be rigorously expressed as

\[ G(\varepsilon_{ij}, H_i) = \frac{1}{2V} \int_{V} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \, dV, \]

\[ C_{ijkl} = C_{ijkl}^0 + \sum_{n=1}^{N} f_n \left[ \varepsilon^{(n)}_{ijkl} \varepsilon^{(n)}_{kl} - \delta^{(n)}_{ijkl} H_k^{(n)} \right], \]

where \( V_n = \frac{fn}{V} \) is the volume of the \( n \)th inhomogeneity. The effective moduli can be expressed by substituting Eqs. (78) and (79) into Eq. (75) as

\[ G(\varepsilon_{ij}, H_i) = \frac{1}{2V} \int_{V} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \, dV. \]

To obtain the effective properties of the piezomagnetic material, it is assumed that the far fields with a uniform strain \( \varepsilon^0 \) and a magnetic field \( H^0 \) are applied. According to the thermodynamics, the Gibbs free energy can be expressed by

\[ G(\varepsilon_{ij}, H_i) = G^1(\varepsilon_{ij}, H_i) - G^2(\varepsilon_{ij}, H_i), \]

where

\[ G^1(\varepsilon_{ij}, H_i) = \frac{1}{2V} \int_{V} \varepsilon_{ij} \sigma_{ij}(\varepsilon_{ij}, H_i) \, dV, \]  

\[ G^2(\varepsilon_{ij}, H_i) = \frac{1}{2V} \int_{V} H_i B_i(\varepsilon_{ij}, H_i) \, dV. \]

Through the quasi-static equilibrium equation, the Gauss’s law, and several integrating, Eqs. (76) and (77) can be expressed, respectively, as

\[ G^1(\varepsilon_{ij}, H_i) = \frac{1}{2} \int_{V} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \, dV, \]

\[ C_{ijkl} = C_{ijkl}^0 + \sum_{n=1}^{N} f_n \left[ \varepsilon^{(n)}_{ijkl} \varepsilon^{(n)}_{kl} - \delta^{(n)}_{ijkl} H_k^{(n)} \right], \]

where \( V_n = \frac{fn}{V} \) is the volume of the \( n \)th inhomogeneity. The effective moduli can be expressed by substituting Eqs. (78) and (79) into Eq. (75) as

\[ G(\varepsilon_{ij}, H_i) = \frac{1}{2V} \int_{V} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \, dV. \]

By substituting Eqs. (78) and (79) into Eq. (75), Eq. (75) can be rewritten as

\[ G(\varepsilon_{ij}, H_i) = \frac{1}{2} \int_{V} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \, dV, \]

where \( V_n = \frac{fn}{V} \) is the volume of the \( n \)th inhomogeneity. The effective moduli can be expressed by substituting Eqs. (78) and (79) into Eq. (75) as

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where \( V_n = \frac{fn}{V} \) is the volume of the \( n \)th inhomogeneity. The effective moduli can be expressed by substituting Eqs. (78) and (79) into Eq. (75) as

\[ G(\varepsilon_{ij}, H_i) = \frac{1}{2} \int_{V} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \, dV. \]

Eq. (82) describes the rigorous relation between the effective moduli and the properties of the matrix and inhomogeneities through the effective energy of the piezomagnetic composite. The different effective moduli of the composite can be obtained by the different approximations (such as the dilute solution, the self-consistent approach, Mori–Tanaka method or/and differential scheme). When one of the approximations is applied, \( \varepsilon^{(n)}_{ij} \) and \( H^{(n)}_i \) can be expressed in the general form as

\[ \varepsilon^{(n)}_{ij} = m^{(n)}_{ijkl} \varepsilon^0_{kl} + n^{(n)}_{ijkl} H^0_k, \]

\[ H^{(n)}_i = p^{(n)}_{ikl} \varepsilon^0_{kl} + t^{(n)}_{ij} H^0_j, \]

where \( m^{(n)}_{ijkl}, n^{(n)}_{ijkl}, p^{(n)}_{ikl}, \) and \( t^{(n)}_{ij} \) are relative to the applied approximation and the properties of the matrix and the \( n \)th inhomogeneity. The effective moduli can be expressed by substituting Eqs. (83) and (84) into Eq. (82) as
\[
\bar{C}_{ijkl} = C_{ijkl} + \sum_{n=1}^{N} f_n \text{sym}\left[ e_{ijpq}^{(n)} m_{pqkl}^{(n)} - e_{ijpq}^{(n)} p_{qkl}^{(n)} \right]. \tag{85}
\]

\[
\bar{\mu}_{ij} = \mu_{ij} + \sum_{n=1}^{N} f_n \text{sym}\left[ d_{ik}^{(n)} n_{kl}^{(n)} + k_{ik}^{(n)} f_{kj}^{(n)} \right]. \tag{86}
\]

\[
\bar{d}_{ijk} = \frac{1}{2} \sum_{n=1}^{N} f_n \left[ d_{ijqk}^{(n)} - e_{ijpq}^{(n)} r_{k}^{(n)} + r_{it}^{(n)} a_{tjk}^{(n)} + d_{istj}^{(n)} m_{stjk}^{(n)} \right]. \tag{87}
\]

where “\text{sym } A” denotes the symmetric part of the tensor \( A \), for instance, \( \text{sym } A_{ij kl} = \frac{1}{2} (A_{ij kl} + A_{klij}) \).

4.1. The dilute solution

The dilute approximation is the simplest micromechanical approach assuming that the interaction among the reinforcing inclusions in a matrix-based composite can be ignored. When the strain and the magnetic field in an inclusion embedded in an infinite matrix substitute, respectively, for the mean strain and the mean magnetic field in the inhomogeneity, the dilute solution for the effective moduli can be obtained. Then,

\[
m_{ijkl}^{(n)}, n_{kl}^{(n)}, k_{ij}^{(n)}, p_{ijkl}^{(n)}, r_{ij}^{(n)}, \quad \text{and } \quad r_{ij}^{(n)} \text{ can be expressed as}
\]

\[
m_{ijkl}^{(n)} = N_{ijkl}^{1(n)} + N_{ijkl}^{2(n)} \bar{\varepsilon}_{m}^{kij} + N_{ijkl}^{3(n)} \bar{H}_{j}^{3}, \tag{88}
\]

where \( N_{ijkl}^{1(n)}, N_{ijkl}^{2(n)}, N_{ijkl}^{3(n)} \), and \( N_{ijkl}^{4(n)} \) are the magnetoelastic Eshelby tensors. Substituting Eq. (88) into Eqs. (85)–(87) can yield the effective magnetoelastic moduli.

4.2. The Mori–Tanaka solution

Substituting the mean strain and the mean magnetic field, \( \bar{\varepsilon}_{m}^{m} \) and \( \bar{H}_{m}^{m} \) for \( \varepsilon^{0}_{ij} \) and \( H^{0}_{i} \), into Eqs. (33) and (34) yields the strain and magnetic field in the \( n \)th inhomogeneity with respect to the interaction among the inclusions. So Eqs. (33) and (34) can be rewritten as

\[
\varepsilon_{ij}^{(n)} = N_{ijkl}^{1(n)} \bar{\varepsilon}_{m}^{ij} k_{ij}^{(n)} + N_{ijkl}^{2(n)} \bar{H}_{k}^{m}, \tag{89}
\]

\[
H_{i}^{(n)} = N_{ijkl}^{3(n)} \bar{\varepsilon}_{m}^{ij} k_{ij}^{(n)} + N_{ijkl}^{4(n)} \bar{H}_{j}^{m}, \tag{90}
\]

where

\[
\bar{\varepsilon}_{m}^{ij} = \frac{1}{V_{m}} \int_{V_{m}} \varepsilon_{ij} dV, \tag{91}
\]

\[
\bar{H}_{m}^{m} = \frac{1}{V_{m}} \int_{V_{m}} H_{i} dV \tag{92}
\]

in which \( V_{m} \) is the volume of the matrix. Following a complicated manipulation, the \( m_{ijkl}^{(n)}, n_{kl}^{(n)}, k_{ij}^{(n)}, p_{ijkl}^{(n)}, r_{ij}^{(n)} \) can be expressed as

\[
m_{ijkl}^{(n)} = N_{ijpq}^{1(n)} M_{pqkl}^{ijkl} + N_{ijpq}^{2(n)} p_{ijkl}^{ijkl}, \tag{93}
\]

\[
n_{ijkl}^{(n)} = N_{ijpq}^{1(n)} N_{pqkl}^{ijkl} + N_{ijpq}^{2(n)} q_{pk}, \tag{94}
\]

\[
p_{ijkl}^{(n)} = N_{ijpq}^{3(n)} M_{pqkl}^{ijkl} + N_{ijpq}^{4(n)} p_{ijkl}^{ijkl}, \tag{95}
\]

\[
r_{ij}^{(n)} = N_{ipq}^{3(n)} N_{pqij}^{ijkl} + N_{ipq}^{4(n)} q_{ij}, \tag{96}
\]

where

\[
M_{ijkl} = \left[ H_{ijkl}^{1} - H_{ijkl}^{2} (H_{pqk}^{4})^{-1} H_{ijkl}^{3} \right]^{-1}, \tag{97}
\]

\[
N_{ijkl} = -M_{ijkl} H_{ijkl}^{2} (H_{ijkl}^{4})^{-1}, \tag{98}
\]

\[
P_{ijkl} = -(H_{ijkl}^{4})^{-1} H_{ijkl}^{3} p_{ijkl}^{ijkl}, \tag{99}
\]

\[
Q_{ij} = (H_{ijkl}^{4})^{-1} \left[ \delta_{ki} - H_{ijkl}^{3} N_{ijkl} \right]. \tag{100}
\]
\[ H_{ijkl}^1 = \delta_{ijkl} - \sum_{n=1}^{N} f_n \left[ \delta_{ijkl} - N_l^{3(1)} \right], \]  
(101)  
\[ H_{ijk}^2 = \sum_{n=1}^{N} f_n \kappa_{ij}^{2(n)}, \]  
(102)  
\[ H_{ijk}^3 = \sum_{n=1}^{N} f_n \kappa_{ij}^{3(n)}, \]  
(103)  
\[ H_{ij}^4 = \delta_{ij} - \sum_{n=1}^{N} f_n \left[ \delta_{ij} - H_{ij}^{4(n)} \right]. \]  
(104)

Substituting Eqs. (93)–(96) into (85)–(87) yields the effective moduli of the composite.

5. Results and discussions

As an example, a ferromagnetic composite with the infinitely long circular piezomagnetic inhomogeneities embedded in a non-piezomagnetic matrix is analyzed in detail. The numerical examinations are conducted for BaTiO\textsubscript{3}–CoFe\textsubscript{2}O\textsubscript{4} composite, where the inclusion is the piezomagnetic CoFe\textsubscript{2}O\textsubscript{4}, and the matrix is non-piezomagnetic BaTiO\textsubscript{3}. In fact, BaTiO\textsubscript{3} is piezoelectric, but the magneto-electro-coupling usually can be neglected. Furthermore, both the matrix and the inclusion are considered to be transversely isotropic. The material constants for BaTiO\textsubscript{3} and CoFe\textsubscript{2}O\textsubscript{4} are listed in Table 1 (Huang, 1997, 1998). The predicted overall magneto-mechanical properties are presented in Figs. 1–8.

In Figs. 1–8 illustrate the effect of the volume fraction of the inhomogeneities on the resulting composite properties are obvious. Figs. 1–3 show the predicted elastic constants against the volume fraction of the inhomogeneities. The overall elastic properties, \( \overline{C}_{11}, \overline{C}_{12} \) and \( \overline{C}_{33} \) increase with the increase of the volume fraction, and the Mori–Tanaka predictions are in a close agreement with the dilute prediction at lower volume fractions \( f < 0.23 \). Figs. 4–5 show the predicted magnetic permeabilities against the volume fraction of the inhomogeneities. The predicted piezomagnetic constant, \( \mu_{11} \), decreases with the increase of the volume fraction while \( \mu_{33} \) increases with the increase of the volume fraction. Moreover, the curves predicted by the Mori–Tanaka predictions for \( \mu_{11} \) and \( \mu_{33} \) almost overlap those in the dilute predictions at any volume fraction. Figs. 6–8 show the predicted piezomagnetic coefficients against the volume fraction of the inhomogeneities. In Figs. 6 and 7, the Mori–Tanaka

<table>
<thead>
<tr>
<th>Material</th>
<th>( C_{11} ) (GPa)</th>
<th>( C_{12} ) (GPa)</th>
<th>( C_{13} ) (GPa)</th>
<th>( C_{33} ) (GPa)</th>
<th>( d_{31} ) (N/Am)</th>
<th>( d_{33} ) (N/Am)</th>
<th>( d_{15} ) (N/Am)</th>
<th>( \mu_{11} ) (Ns(^2)/C(^2))</th>
<th>( \mu_{33} ) (Ns(^2)/C(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>BaTiO\textsubscript{3}</td>
<td>166</td>
<td>77</td>
<td>78</td>
<td>162</td>
<td>43</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5.0E–6</td>
</tr>
<tr>
<td>CoFe\textsubscript{2}O\textsubscript{4}</td>
<td>286</td>
<td>173</td>
<td>170.5</td>
<td>269.5</td>
<td>45.3</td>
<td>580.3</td>
<td>699.7</td>
<td>550</td>
<td>-590E–6</td>
</tr>
</tbody>
</table>

Fig. 1. The predicted composite elastic constants \( \overline{C}_{11} \) against volume fraction \( f \).

Fig. 2. The predicted composite elastic constants \( \overline{C}_{12} \) against volume fraction \( f \).
predictions are in close agreement with the dilute prediction only at low volume fractions \((f < 0.2)\), but in Fig. 8, the Mori–Tanaka prediction is the same as the dilute prediction. In general, it is at the value of \(f = 0.2\), the Mori–Tanaka predictions and the dilute predictions begin to significantly diverge each other. It is expected that the Mori–Tanaka predictions will be better than the dilute predictions at higher volume fractions.
6. Concluding remarks

Based on such a consideration that a piezomagnetic ellipsoidal inhomogeneity is embedded in a non-piezomagnetic matrix, two different approaches are employed to the magneto-mechanical inclusion problem: one is the Green’s function method and the other is the equivalent inclusion method. However, For the piezomagnetic composites with a non-piezomagnetic matrix, two closed-form solutions obtained, respectively, by use of these two methods are completely equivalent each other. Furthermore, explicit expressions have been obtained for a set of magnetoelastic Eshelby tensors. The analytical models are proposed by a unified energy method to predict the effective magnetoelastic moduli. The dilute and Mori–Tanaka schemes are discussed, respectively. Finally, the calculated results made for a BaTiO₃–CoFe₂O₄ composite show that the overall elastic properties, \( \overline{C}_{11}, \overline{C}_{12} \) and \( \overline{C}_{33} \) increase with the increase of the volume fraction, and the Mori–Tanaka predictions are in a close agreement with the dilute prediction at lower volume fractions. However, the predicted piezomagnetic constant, \( \mu_{11} \), decreases with the increase of the volume fraction while \( \mu_{33} \) increases with the increase of the volume fraction. In general, it is at the value of \( f = 0.2 \), the Mori–Tanaka predictions and the dilute predictions begin to significantly diverge each other. It is expected that the Mori–Tanaka predictions will be better than the dilute predictions at higher volume fractions.

Acknowledgement

The authors are grateful for the support by National Natural Science Foundation of China under grants #10025209, #10132010, #10102007, #90208002. Supported by the Key grant Project of Chinese Ministry of Education (0306) is also acknowledged.

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